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Dynamics of backreactions around pure de Sitter spacetime

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Abstract

The *leading* order gravitational effect of nonlinear matter fluctuations in de Sitter spacetime is briefly analysed and it is found that the initial value problem for the perturbed Einstein equations possesses so-called linearization instabilities. Using previous results, I very briefly review how these linearization instabilities can be avoided by assuming strict de Sitter invariance of the quantum states of the linearized fluctuations. I then sketch how quantum anomalies do not block the invariance requirement. Finally, I explore tantalizing evidence that the same stringent symmetry requirement may also be present when gravitational *wave* backreactions are included.

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1. Introduction

The relative importance of nonlinear quantum fluctuations in contemporary cosmological models is still not well understood, not least because the theoretical tools to understand interacting quantum fields in curved spacetime are only now emerging in rigorous form. In previous work [1], Unruh and I used some of these new technical developments (see e.g. [3] and [6]) in quantum field theory to explore the consequences of coupling fluctuations in a scalar field (which naturally arise at *second* order in perturbation theory) to the *leading* order fluctuations of the gravitational field, which in the background we took as the well-known de Sitter solution. We found that the requirement that these fluctuations solve the Einstein equations at second order in perturbation theory severely restricts the possible symmetry their quantum states can possess, and that, interestingly, quantum anomalies arising from the renormalization freedom, one has in the problem, do not ruin this restriction. In this paper I survey these results with an eye to generalizing them. In particular I show in what follows that de Sitter invariance likely persists when certain (subleading) gravitational couplings, which were ignored in our previous calculation, are included.

2. 'Naive' perturbative approximation

We begin by perturbing the usual Einstein field equations

$$G_{ab}(g_{ab}) = \kappa T_{ab}(g_{ab}, \phi), \tag{1}$$

where $\kappa \equiv 8\pi G$, about the de Sitter solution

$$\mathrm{d}s^2 = \bar{g}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = -\mathrm{d}t^2 + \cosh(t)^2(\mathrm{d}\chi^2 + \sin(\chi)^2 d\Omega(\theta,\eta)^2) \tag{2}$$

to second order in a small parameter ϵ. Here t, χ, θ, η are the usual coordinates of the closed chart covering de Sitter via $\Re \times S^3$, where the comoving time is t and the angles (χ, θ, η) represent the angles of S^3, and $\mathrm{d}\Omega^2$ is the S^2 metric $\mathrm{d}s^2 = d\theta^2 + \sin(\theta)^2 d\eta^2$. We write our perturbation ansatz as

$$g_{ab} = \bar{g}_{ab}(t,\chi,\theta,\eta) + \epsilon^2\delta^2 g_{ab}(t,\chi,\theta,\eta) \tag{3}$$

$$\phi = \bar{\phi} + \epsilon\delta\phi(t,\chi,\theta,\eta),\ \bar{\eta} \in \Re,\ \bar{\phi} \in \Re, \tag{4}$$

i.e. the leading order gravitational fluctuations are taken to occur at *second* order in ϵ, with background quantities denoted by overbars, and the scalar field is denoted by ϕ.

Working within this approximation, however, the leading order gravitational perturbation of the de Sitter background can be symbolically written as

$$\underbrace{\mathcal{L}[\delta^2 g_{ab}]}_{\text{Linearized gravity}} = \underbrace{\kappa\mathcal{Q}_{ab}[(\delta\phi)(\delta\phi)]}_{\text{Nonlinear source}}, \tag{5}$$

where $\mathcal{L}$ is a *linear* second order hyperbolic operator and $\mathcal{Q}_{ab}$ is an operator of mixed character acting on the quadratic collection of matter fluctuations. We furthermore assume that the fluctuations $\delta\phi$ are *massless*, i.e. that they satisfy

$$\bar{\Box}\delta\phi = 0, \tag{6}$$

where $\bar{\Box} \equiv \bar{\nabla}^c\bar{\nabla}_c$. One can show that

$$\mathcal{L}[\delta^2 g_{ac}] = \left(\bar{\Box} + \frac{2\Lambda}{3}\right)\delta^2 g_{ac} + \left(\frac{\Lambda\bar{g}_{ac}}{3} - \bar{\nabla}_a\bar{\nabla}_c + \bar{\Box}\bar{g}_{ac}\right)\delta^2 g - \bar{g}_{ac}\bar{\nabla}^\ell\bar{\nabla}^m\delta^2 g_{\ell m} + 2\bar{\nabla}_{(a}\bar{\nabla}^m\delta^2 g_{c)m}, \tag{7}$$

where $\delta^2 g \equiv \bar{g}^{ab}\delta^2 g_{ab}$ is the trace of the metric perturbation and where the standard summation convention applies for repeated indices and index symmetrization. The right-hand side of equations (5) is simple enough to derive by inspection of equation (1). The result is

$$\kappa\mathcal{Q}_{ac}[(\delta\phi)(\delta\phi)] = 2\kappa\left(\bar{\nabla}_a\delta\phi\bar{\nabla}_c\delta\phi - \frac{\bar{g}_{ac}}{2}\bar{\nabla}_m\delta\phi\bar{\nabla}^m\delta\phi\right), \tag{8}$$

and thus equations (7) and (8) combined in equations (5), along with the matter equation (6), form the combined second order Einstein-matter equations we wish to solve.

2.1. *Quantum anomalies*

Formally, all of the above expressions containing 'Wick monomials' $(\delta\phi)^2$ and $\delta\phi\bar{\nabla}_a\bar{\nabla}_b\delta\phi$ are intrinsically meaningless without some renormalization or regulation to treat the infinities, since the operator $\delta\phi$ is a distribution. After applying a given renormalization scheme it is likely that not all of the gauge conditions, equations of motion, or any further conditions, can hold simultaneously *especially* if one furthermore demands that they be local or covariant in the sense of Hollands and Wald (HW) in [3]. For example, HW prove that for a massless, free quantum field $\delta\phi$ satisfying the linear equation of motion

$$\bar{\Box}\delta\phi = 0 \tag{9}$$

it is *not* in general possible to *also* satisfy the nonlinear conditions

$$\delta\phi\bar{\Box}\delta\phi = 0 \tag{10}$$

$$(\bar{\nabla}_b\delta\phi)\bar{\Box}\delta\phi = 0 \tag{11}$$

if one insists on $\delta\phi$ being a local and covariant quantum field. In this sense a quantum anomaly is said to occur. However, as I showed with Bill Unruh in [1], the conditions HW derived which forbid the simultaneous satisfaction of the auxiliary conditions (10) and (11) (which are important in key gauge-fixings, see [1]) with the equation of motion are actually satisfied for a de Sitter background spacetime.

We may now ask if it is possible to impose additional conditions, in particular the so-called linearization stability conditions, in a consistent way.

2.2. Linearization stability

The issue of whether or not a given solution of the linearized field equations actually represents a linearization of an exact solution is encapsulated by the concept of 'linearization stability', which was discovered in the 1970s by various people (see e.g. [2]) for spatially closed spacetimes with Killing isometries. To guarantee this stability, the fluctuations obey certain nonlinear conditions.

As shown in [1], when one projects the constraints $\delta^2 P(X^a)$ along a general Killing vector $X^a = (X_\perp, X^i)$ the result can be written as

$$\begin{aligned}\delta^2 P(X^a) \equiv 2\kappa \int_{S^3} \sqrt{|\bar{h}|}\,\Bigg\{ &\left[\frac{1}{2}\bar{D}^i\bar{D}_i\Psi - \bar{h}^{ij}\Psi_{ij} - \frac{3H}{2}(\partial_a\Psi)\bar{n}^a\right] X_\perp \\ &+ \frac{1}{2}\left[-\Psi_{ab} + \frac{1}{2}\partial_a\partial_b\Psi\right] X_\perp \bar{n}^a\bar{n}^b + \left[\left(\Psi_{ia} - \bar{n}_a\frac{3H}{2}\bar{\nabla}_i\Psi\right) X^i \right.\\ &\left. + \frac{1}{2}(\bar{\nabla}_a\Psi)\bar{D}_i X^i\right]\bar{n}^a\Bigg\}\, d^3x \overset{*}{=} 0,\end{aligned} \tag{12}$$

where $H \equiv \partial_0 \ln(a(t))$ and $3H^2 = \kappa\Lambda$, $\Psi \equiv (\delta\phi)^2$, $\Psi_{ab} \equiv \delta\phi\bar{\nabla}_a\bar{\nabla}_b\delta\phi$, and $\bar{D}_a, \bar{n}^a$ are the background spatial covariant derivative and normal vector on S^3 (with metric $\bar{h}_{ab}$) respectively. In this case the LS conditions constrain the matter fluctuations $\delta\phi$ alone; however, in the next part we briefly indicate how a generalization to include the gravitational waves may be achieved.

2.3. Quantum anomalies in the LS conditions; de Sitter invariance

As we show in [1], the quantum anomalies in the LS conditions (plus the equations of motion and other conditions) are proportional to an integral over S^3 of the normal component $X_\perp$ of the given Killing vector, which is zero by an identity. We concluded that the LS conditions (12) do *not* exhibit any quantum anomalies with respect to the given coordinate conditions, the equations of motion, and the requirements of locality and covariance in the sense of HW. They do form a nontrivial operator constraint on the quantum states $|\Phi\rangle$ which the operators $\delta\phi$ act on, and we furthermore showed that they generate the de Sitter ($SO(4,1)$) transformations, which means they demand manifest de Sitter invariance of all states $|\Phi\rangle$ (see Moncrief in [2], and Higuchi in [4, 5]).

3. Generalization of perturbative approximation

It is clear that ignoring the gravitational wave perturbations and assuming an inert linear gauge sector, in the ansatz of equation (3) of section 2, represents a serious approximation and a possible loophole to our main result. It is not at all clear that the constraints $\delta^2 P(X^a)$ will still have the algebra of the $SO(4,1)$ generators if we include the TT–TT contributions since these terms may introduce new anomalies. However, the maximal symmetry of the de Sitter background spacetime may be forgiving even in this case. Indeed, consider the more general perturbative ansatz

$$g_{ab} = \bar{g}_{ab}(t,\chi,\theta,\phi) + \epsilon\delta g_{ab}(t,\chi,\theta,\eta) + \epsilon^2\delta^2 g_{ab}(t,\chi,\theta,\eta) \tag{13}$$

$$\phi = \bar{\phi} + \epsilon\delta\phi(t,\chi,\theta,\eta), \quad \bar{\eta}\in\Re, \quad \bar{\phi}\in\Re, \tag{14}$$

which incidentally is self-consistent since to second order the stress energy cannot contain any $\delta^2\phi$ terms. It is well known that there exists a gauge fixing such that the only physical degrees of freedom in the gravitational sector, to linear order in ϵ, are transverse traceless excitations (i.e. there are no physical scalar modes, as one would think reasonable since de Sitter is a vacuum solution ($\bar{\phi}\in\Re$)) with no 'shift' or 'lapse' perturbations, i.e. 'linear gravitational waves'.

It was famously shown by Parker and Ford in [7] that quantizing linear gravitational wave perturbations in spatially flat FRW universes is *equivalent* to quantizing a pair of massless, minimally coupled scalar fields in the same background spacetime. In de Sitter spacetime one can show that there exists a coordinate system (for the global $\Re\times S^3$ patch) in which the linearized Einstein equation is

$$\left(\bar{\nabla}^m\bar{\nabla}_m - \frac{2\Lambda}{3}\right)\delta g_{ac} = 0, \tag{15}$$

and the conditions $\bar{g}^{ac}\delta g_{ac} = 0$, $\bar{\nabla}^m\delta g_{am} = 0$, and *also* $\delta g_{0a} = 0$ hold. One may expand the perturbation δg_{ac} in terms of the basis functions on S^3, i.e. write $\delta g_{ab} \equiv A(t)\Psi_{ab}(\chi,\theta,\phi)$ such that the generalized spherical harmonics on S^3 obey the (discrete) Laplace–Beltrami eigenvalue condition (see [8] for more detail) for each ℓ mode

$$\bar{\nabla}^i\bar{\nabla}_i\Psi^{(\ell)}_{ab} \equiv \bar{\Delta}_{S^3}\Psi^{(\ell)}_{ab} = \frac{2\Lambda}{6}(-\ell(\ell+2)+2)\Psi^{(\ell)}_{ab}, \tag{16}$$

where $\ell\geqslant 2$ is an integer. Inserting this decomposition into the field equations (15) yields (using equation (2)), for $\ell\geqslant 2$ an integer,

$$\frac{1}{\cosh(t)}\partial_t[\cosh^3(t)\partial_t A(t)] + \frac{2\Lambda}{6}[-\ell(\ell+2)+2]A(t) - \frac{2\Lambda}{3}A(t) = 0, \tag{17}$$

from which it is clear that the last two terms cancel completely. However, as one may easily verify, this is just the equation for a massless minimally coupled scalar field in de Sitter spacetime *if* $\ell = 0, 1$ are allowed. Therefore all we can prove here is that the two independent degrees of freedom of the linearized gravitational field (polarizations) in de Sitter spacetime merely have the same *dynamics* as two minimally coupled *scalar* fields[1].

3.1. The $\ell = 0, 1$ scalar modes

It is physically clear that the $\ell = 0, 1$ modes in the scalar $\delta\phi$ sector do not, by definition, have any analogues in the gravitational sector δg_{ab} for a spatially closed universe. However one

[1] Presumably this weaker statement of Parker and Ford's result is true for all conformally flat background spacetimes.

may now be able to use the fact that there are two gauge sectors in the problem, a linear one used to effect spatial transverse tracelessness and a thus far *free* second order sector, to remove the $\ell = 0, 1$ modes altogether from the scalar sector. Note though that by definition any mode decomposition of a quantum field is of a global nature so it is difficult to see *a priori* how any such gauge condition can be local and covariant in the sense of HW. Nevertheless, consider the leading order gauge transformation of $\delta\phi$,

$$\delta\tilde{\phi}' = \delta\phi + 2\pounds_{\bar{\zeta}}\delta\phi + 3\pounds_{\bar{\chi}}\delta\phi \tag{18}$$

where ζ^a already effects conditions which render δg_{ab} transverse, traceless and spatial in our coordinates[2] and χ^a is the second order infinitesimal transformation. It can be straightforwardly shown that the identity

$$\Lambda\pounds_{\zeta}\delta\phi = \Lambda\zeta^a\bar{\nabla}_a\delta\phi = 0 \tag{19}$$

holds classically in de Sitter spacetime, given only the equations of motion of $\delta\phi$, the conditions on ζ^a and maximal symmetry, and similarly that there are no quantum anomalies (in the sense of HW) introduced by them[3]. Furthermore it is straightforward to demonstrate that one can construct a second order gauge fixing by solving the following conditions:

$$\Lambda\bar{\nabla}^a({}^{(2)}T\bar{\nabla}_a\delta\phi) = F_1(\Lambda)\delta\phi \tag{20}$$

$$\Lambda^{(2)}L^a\bar{\nabla}_a\delta\phi = F_2(\Lambda)\delta\phi, \tag{21}$$

where F_1 and F_2 are nontrivial, unique, linear combinations of Λ, and *defining* the vector field $\chi^a \equiv \bar{\nabla}^{a\,(2)}T + {}^{(2)}L^a$ using the solutions $({}^{(2)}T, {}^{(2)}L^a)$. It can be shown that the resulting χ^a can be used in equation (18) to ensure that the $\ell = 0, 1$ scalar modes are gauged away. One can also show, similarly to the identity (19), that there are no additional quantum anomalies introduced by insisting equations (20) and (21) hold. Thus, it appears possible to sensibly reduce the space of solutions for the scalar sector to match that of the gravitational sector by exploiting part of the available second order gauge freedom in the problem.

3.2. LS conditions and de Sitter invariance

Formally, we may use the above result to write the gravitational wave stress energy as that of a pair of minimally coupled scalar fields. In this case the final LS conditions, equations (12) would be modified only by writing $\Psi \rightarrow \Psi + \Psi(h_+) + \Psi(h_\times)$ and $\Psi_{ab} \rightarrow \Psi_{ab} + \Psi_{ab}(h_+) + \Psi_{ab}(h_\times)$, where h_+ and $h_\times$ are the suitably defined polarization (scalar) fields of the gravitational field. It is clear that under this reidentification of Ψ and Ψ_{ab} the results will still hold and that, in particular, the requirement of de Sitter invariance will not be ruined given the previous arguments (in particular the arguments around equations (20) and (21)).

4. Conclusions

While the linear matter (scalar field) theory on a background de Sitter spacetime is well defined and has a rich set of possible states, it is interesting to observe what the effects that any nonlinearities, and in particular the coupling of the scalar field to gravity, are on the theory. The surprising result is that the very requirement that one be able to couple

[2] Of course, equation (18) will in general involve terms like $\pounds^2_{\zeta}\bar{\phi}$, $\pounds_{\chi}\bar{\phi}$, etc for $\bar{\phi}$ nontrivial.

[3] Essentially because it is a 'derivative condition' on any anomaly, which for maximal symmetry is a spacetime constant.

the scalar field (in leading order) to gravity in and of itself places severe constraints on the allowed states for the scalar field. In this short paper I explored evidence that these severe constraints, namely manifest de Sitter invariance, persist even when subleading, gravitational *wave* backreactions, are allowed. I found that it seems possible to circumvent a complicated generalization of Hollands' and Wald's work to spin-2 interactions and use an observation due to Parker and Ford to essentially reduce the problem to one with interacting *scalar* fields in de Sitter spacetime.

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